

COHOMOLOGY OF SPLIT GROUP EXTENSIONS AND CHARACTERISTIC CLASSES

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ABSTRACT. There are characteristic classes that are the obstructions to the vanishing of the differentials in the Lyndon-Hochschild-Serre spectral sequence of a split extension of an integral lattice L by a group G . These characteristic classes exist in the r -th page of the spectral sequence provided that the differentials $d_i = 0$ for all $i < r$. When L decomposes into a sum of G -sublattices, we show that there are defining relations between the characteristic classes of L and the characteristic classes of its summands.

1. INTRODUCTION

Suppose G is a group and L is a finite rank integral $\mathbb{Z}G$ -lattice. Let $\Gamma = L \rtimes G$ be the semidirect product group induced by the action of G on L . Our objective is to analyze the split group extension $0 \rightarrow L \rightarrow \Gamma \rightarrow G \rightarrow 1$ and its associated Lyndon-Hochschild-Serre spectral sequence $\{E_*, d_*\}$. In [1] we showed that when G is a cyclic group of prime order, the spectral sequence with integral coefficients collapses at E_2 without extension problems. In fact, our proof stated in [1] applies not only to integral coefficients, but to all coefficient modules in a certain category which we denote by $\mathcal{M}_F(L, G)$. This category is essential in the definition of characteristic classes (see Theorem 2.2). It consists of all finite rank $F\Gamma$ -lattices M on which L acts trivially, where F is a given principal ideal domain. In particular, F as a trivial $F\Gamma$ -module is an object of $\mathcal{M}_F(L, G)$.

Let us now suppose that L decomposes into a sum of $\mathbb{Z}G$ -sublattices; $L = L' \oplus L''$. Let $\{E'_*, d'_*\}$ and $\{E''_*, d''_*\}$ be the Lyndon-Hochschild-Serre spectral sequences associated to the respective semidirect product groups $L' \rtimes G$ and $L'' \rtimes G$. The goal of the present paper is to find necessary and sufficient conditions on the latter defined spectral sequences such that $\{E_*, d_*\}$ has no nonzero differentials up to a given page.

In section 2, we define characteristic classes $v_r^t(L)$, introduced by Charlap-Vasquez and Sah (see [2] and [3]), that are the obstructions to the vanishing of the differentials in $\{E_*, d_*\}$. These classes exist and lie in the images of the differentials $d_r^{0,t} :$

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$E_r^{0,t}(H_t(L, F)) \rightarrow E_r^{r,t-r+1}(H_t(L, F))$ provided that the differentials $d_i^{*,t}$ vanish for all $i < r$ and for all coefficient modules in $\mathcal{M}_F(L, G)$. Our main result is the following decomposition theorem.

Theorem 1.1. Let G be any group. Assume L' and L'' are $\mathbb{Z}G$ -lattices of finite rank and $L = L' \oplus L''$. Let $r \geq 2$ and $t \geq 0$.

- (a) Suppose $v_k^i(L') = v_k^j(L'') = 0$ for all $i, j \leq t$ and for all $k < r$. Then $d_2^{s,m} = \dots = d_{r-1}^{s,m} = 0$ for all $s \geq 0$, all $m \leq t$, and all coefficient modules in $\mathcal{M}_F(L, G)$. Additionally,

$$v_r^t(L) = \sum_{i+j=t} \left((C_j'^r)_*(v_r^i(L')) + (-1)^i (C_i''^r)_*(v_r^j(L'')) \right).$$

- (b) Suppose $v_k^t(L) = 0$ for all $k < r$. Then $d_k^{s,t} = d_k''^{s,t} = 0$ for all $s \geq 0$, all $k < r$, and all coefficient modules in $\mathcal{M}_F(L, G)$. Additionally,

$$v_r^t(L') = D_*'^r(v_r^t(L)) \quad \text{and} \quad v_r^t(L'') = D_*''^r(v_r^t(L)).$$

In section 3, we define the FG -equivariant homomorphisms $C_j'^r$, $C_i''^r$, D'^r , and D''^r that induce the maps $(C_j'^r)_* : E_2'^{s,i-r+1} \rightarrow E_2^{s,t-r+1}$, $(C_i''^r)_* : E_2''^{s,j-r+1} \rightarrow E_2^{s,t-r+1}$, $D_*'^r : E_2^{s,t-r+1} \rightarrow E_2'^{s,t-r+1}$, and $D_*''^r : E_2^{s,t-r+1} \rightarrow E_2''^{s,t-r+1}$, respectively. The homomorphisms $C_j'^2$ and $C_i''^2$ were first considered by Charlap and Vasquez in [2]. They showed that the sum formula holds when $r = 2$. We use a different approach to generalize this result to all pages of the spectral sequence and also to prove a converse.

The problem of establishing the collapse of $\{E_*, d_*\}$ in general can be a difficult one. The spectral sequence can have nonzero differentials even when G is abelian. For instance, Totaro proved in [4] that for any prime number p and $G = C_p^2$, there is a semidirect product group $\Gamma = L \rtimes C_p^2$ such that in the associated Lyndon-Hochschild-Serre spectral sequence with \mathbb{Z}_p coefficients there always exist nonzero differentials at E_p or later.

A question, first posed by Adem (see [1]), that is still open is whether the spectral sequence collapses integrally at E_2 without extension problems when G is an arbitrary finite cyclic group. In view of our results, we can make the following

Conjecture 1.2. Let C_n be a cyclic group of order n and let L be a finite rank integral lattice. The Lyndon-Hochschild-Serre spectral sequence $\{E_*, d_*\}$ of any split group extension $0 \rightarrow L \rightarrow \Gamma \rightarrow C_n \rightarrow 1$ collapses at E_2 for all coefficient modules in $\mathcal{M}_F(L, C_n)$.

Note that part (a) of Theorem 1.1 reduces this conjecture to the case where L is an indecomposable $\mathbb{Z}C_n$ -lattice (see Corollary 4.2).

2. PRELIMINARY RESULTS

Henceforth, let G be any group and let L be any finite rank integral $\mathbb{Z}G$ -lattice. Denote by Γ the associated semidirect product $L \rtimes G$. Suppose F is a principal ideal domain. For each $F\Gamma$ -module M we have the Lyndon-Hochschild-Serre spectral sequence

$$E_2^{p,q}(M) = H^p(G, H^q(L, M)) \implies H^{p+q}(\Gamma, M).$$

In the proof of our main result, we make use of the multiplicative structure of this spectral sequence. Namely, any Γ -pairing of $F\Gamma$ -modules A , B , and C

$$\cdot : A \otimes_F B \rightarrow C,$$

determines an F -pairing

$$E_r^{p,q}(A) \otimes_F E_r^{s,t}(B) \rightarrow E_r^{p+s, q+t}(C).$$

In addition, for the differential d_r and for each $a \in E_r^{p,q}(A)$ and $b \in E_r^{s,t}(B)$ we have the product formula

$$d_r^{p+s, q+t}(a \cdot b) = d_r^{p,q}(a) \cdot b + (-1)^{p+q} a \cdot d_r^{s,t}(b).$$

We denote by $\mathcal{M}_F(L, G)$ the category of all $F\Gamma$ -modules M , such that M is a finitely generated free F -module and L acts trivially on M . In particular, F with a trivial Γ -action is a module in this category. $\mathcal{M}_F(L, G)$ has the property that if $M \in \mathcal{M}_F(L, G)$, then $H_i(L, M)$ and $H^i(L, M) \cong \text{Hom}_F(H_i(L, F), M)$ are also in $\mathcal{M}_F(L, G)$. Observe that the projection of Γ onto G induces an equivalence between this category and the category of all finite rank FG -lattices.

Let $t \geq 0$ and $r \geq 2$. Set $M = H_t(L, F) \cong \wedge^t(L) \otimes F$. Assume $d_k^{0,t} = 0$ for all $2 \leq k \leq r-1$. By applying the Universal Coefficient Theorem it follows

$$\begin{aligned} E_r^{0,t}(H_t(L, F)) &= E_2^{0,t}(H_t(L, F)) \\ &= H^0(G, H^t(L, H_t(L, F))) \\ &\cong H^0(G, \text{Hom}_F(H_t(L, F), H_t(L, F))) \quad (\text{by UCT}) \\ &= \text{Hom}_{FG}(H_t(L, F), H_t(L, F)) \\ &\cong \text{Hom}_{FG}(\wedge^t(L) \otimes F, \wedge^t(L) \otimes F). \end{aligned}$$

Definition 2.1. With the preceding assumptions, let $\text{id}^t : \wedge^t(L) \otimes F \rightarrow \wedge^t(L) \otimes F$ be the identity homomorphism. Identifying along the above isomorphisms, denote by $[f]$ the class in $E_r^{0,t}(H_t(L, F))$ corresponding to a map $f \in \text{Hom}_{FG}(\wedge^t(L) \otimes F, \wedge^t(L) \otimes F)$. Then $v_r^t(L) := d_r^{0,t}([\text{id}^t]) \in E_r^{r,t-r+1}(H_t(L, F))$ is said to be a *characteristic class* of the spectral sequence $\{E_r, d_r\}$.

Characteristic classes were first considered by Charlap and Vasquez in [2], but only in the case when $r = 2$. Sah in [3] extended their definition to all $r \geq 2$ by proving the following key theorem.

Theorem 2.2. (Sah, 1972, [3]) Let $\{E_*, d_*\}$ be the Lyndon-Hochschild-Serre spectral sequence of the extension $0 \rightarrow L \rightarrow \Gamma \rightarrow G \rightarrow 1$. Suppose there exist integers $r \geq 2$ and $t \geq 0$ such that $d_2^{s,t} = \cdots = d_{r-1}^{s,t} = 0$ for all $s \geq 0$ and for all coefficient modules in $\mathcal{M}_F(L, G)$.

(a) There is a canonical epimorphism

$$\theta : E_r^{s,0}(H^t(L, M)) \rightarrow E_r^{s,t}(M) \text{ for all } s \geq 0 \text{ and for all } M \in \mathcal{M}_F(L, G).$$

(b) We have

$$d_r^{s,t}(x) = (-1)^s y \cdot v_r^t(L),$$

for all $x \in E_r^{s,t}(M)$, all $M \in \mathcal{M}_F(L, G)$, and all $y \in E_r^{s,0}(H^t(L, M))$ with $\theta(y) = x$.

(c) Let $\sigma : H \rightarrow G$ be a group homomorphism which converts L into a $\mathbb{Z}H$ -module. Assume $d_2^{s,t} = \cdots = d_{r-1}^{s,t} = 0$ holds for all $s \geq 0$ and for all objects in $\mathcal{M}_F(L, H)$. The characteristic class $w_r^t(L)$ for the category $\mathcal{M}_F(L, H)$ is then the image of $v_r^t(L)$ under the map induced on the spectral sequences.

Note that, with the assumptions of the theorem, characteristic classes $v_r^t(L)$ are obstructions to the vanishing of the differentials $d_r^{s,t}$ for all integers $s \geq 0$. The following corollary is an immediate consequence of Sah's theorem.

Corollary 2.3. Let $t \geq 0$. The Lyndon-Hochschild-Serre spectral sequence $\{E_*, d_*\}$ has $d_r^{s,t} = 0$ for all $s \geq 0$, all $r < n$, and all $M \in \mathcal{M}_F(L, G)$ if and only if the edge differentials $d_r^{0,t} : E_r^{0,t}(H_t(L, F)) \rightarrow E_r^{r,t-r+1}(H_t(L, F))$ are zero for all $r < n$.

Corollary 2.4. Suppose $\varphi : L \rightarrow \Gamma$ is the natural inclusion. Given an integer $t \geq 0$, the following statements are equivalent.

(a) In $\{E_*, d_*\}$, the differentials $d_r^{s,t}$ vanish for all $r, s \geq 0$ and all $M \in \mathcal{M}_F(L, G)$.

- (b) $\varphi^* : H^t(\Gamma, M) \rightarrow H^t(L, M)$ maps onto $H^t(L, M)^G$ for all $M \in \mathcal{M}_F(L, G)$.
(c) $\varphi^* : H^t(\Gamma, H_t(L, F)) \rightarrow H^t(L, H_t(L, F))$ maps onto the G -invariants $H^t(L, H_t(L, F))^G$.

Proof. Clearly (b) implies (c). Note that the map φ^* is given by the composition

$$H^t(\Gamma, M) \twoheadrightarrow E_\infty^{0,t}(M) = E_{t+2}^{0,t}(M) \subset \cdots \subset E_2^{0,t}(M) = H^t(L, M)^G \subset H^t(L, M).$$

If $d_r^{s,t} = 0$ for all $r, s \geq 0$ and all $M \in \mathcal{M}_F(L, G)$, then $E_\infty^{0,t}(M) = E_{t+2}^{0,t}(M) = \cdots = E_2^{0,t}(M)$ for all $M \in \mathcal{M}_F(L, G)$. Hence, (a) implies (b). To prove that (a) follows from (c), assume $\varphi^* : H^t(\Gamma, H_t(L, F)) \rightarrow H^t(L, H_t(L, F))^G$ is onto. Then by the above composition we have $E_\infty^{0,t}(H_t(L, F)) = E_{t+2}^{0,t}(H_t(L, F)) = \cdots = E_2^{0,t}(H_t(L, F))$. This shows that $d_r^{0,t} : E_r^{0,t}(H_t(L, F)) \rightarrow E_r^{r,t-r+1}(H_t(L, F))$ is zero for all $r \geq 2$. By the previous corollary $d_r^{s,t} = 0$ for all $s \geq 0, r \geq 2$, and $M \in \mathcal{M}_F(L, G)$. \square

3. DECOMPOSITION THEOREM

Suppose L is a direct sum of $\mathbb{Z}G$ -sublattices L' and L'' . In this section we derive relations between the characteristic classes of L and the characteristic classes of L' and L'' .

For convenience, we use $\wedge^*(L)$ to denote $\wedge^*(L) \otimes F$. Recall that there is a standard FG -module decomposition

$$\wedge^n(L) \cong \bigoplus_{i+j=n} \wedge^i(L') \otimes \wedge^j(L'').$$

Definition 3.1. Given integers $i, j \geq 0$ and $r \geq 1$, define FG -equivariant homomorphisms

$$C_j'^r : \text{Hom}_F(\wedge^i(L'), \wedge^{i+r-1}(L')) \rightarrow \text{Hom}_F(\wedge^{i+j}(L), \wedge^{i+j+r-1}(L))$$

by

$$C_j'^r(f)(x \otimes y) = \begin{cases} f(x) \otimes y & \text{if } x \in \wedge^i(L') \text{ and } y \in \wedge^j(L'') \\ 0 & \text{otherwise} \end{cases}$$

and

$$C_i''^r : \text{Hom}_F(\wedge^j(L''), \wedge^{j+r-1}(L'')) \rightarrow \text{Hom}_F(\wedge^{i+j}(L), \wedge^{i+j+r-1}(L))$$

by

$$C_i''^r(f)(x \otimes y) = \begin{cases} x \otimes f(y) & \text{if } x \in \wedge^i(L') \text{ and } y \in \wedge^j(L'') \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Gamma' = L' \rtimes G$ and $\Gamma'' = L'' \rtimes G$. There are associated split exact sequences of groups

$$0 \rightarrow L' \rightarrow \Gamma' \rightarrow G \rightarrow 1 \quad \text{and} \quad 0 \rightarrow L'' \rightarrow \Gamma'' \rightarrow G \rightarrow 1.$$

Suppose $v_*^*(L')$ and $v_*^*(L'')$ are the respective characteristic classes of the Lyndon-Hochschild-Serre spectral sequences $\{E'_*, d'_*\}$ and $\{E''_*, d''_*\}$ corresponding to these extensions. Let $\iota' : L' \rightarrow L$ be the natural inclusion and let $p' : L \rightarrow L'$ be the natural projection. Similarly, define $\iota'' : L'' \rightarrow L$ and $p'' : L \rightarrow L''$. As a straightforward application of the definitions, we obtain the following lemma.

Lemma 3.2. The map $C_0'^r : \text{Hom}_F(\wedge^i(L'), \wedge^{i+r-1}(L')) \rightarrow \text{Hom}_F(\wedge^i(L), \wedge^{i+r-1}(L))$ is given by the composition $p'^* \circ \iota'_*$, where

$$\text{Hom}_F(\wedge^i(L'), \wedge^{i+r-1}(L')) \xrightarrow{\iota'_*} \text{Hom}_F(\wedge^i(L'), \wedge^{i+r-1}(L)) \xrightarrow{p'^*} \text{Hom}_F(\wedge^i(L), \wedge^{i+r-1}(L)),$$

and $C_0''^r$ is the composition $p''^* \circ \iota''_*$, where

$$\text{Hom}_F(\wedge^j(L''), \wedge^{j+r-1}(L'')) \xrightarrow{\iota''_*} \text{Hom}_F(\wedge^j(L''), \wedge^{j+r-1}(L)) \xrightarrow{p''^*} \text{Hom}_F(\wedge^j(L), \wedge^{j+r-1}(L)).$$

Proof. Suppose $f \in \text{Hom}_F(\wedge^i(L'), \wedge^{i+r-1}(L'))$. For any $x \in \wedge^i(L')$ and $y \in \wedge^0(L'')$, $(p'^* \circ \iota'_*)(f)(x \otimes y) = y(\iota'_*(f)(x)) = y(f(x) \otimes 1) = f(x) \otimes y = C_0'^r(f)(x \otimes y)$. The second assertion of the lemma follows analogously. \square

Definition 3.3. Given integers $i, j \geq 0$ and $r \geq 1$, let $\wedge^{i+r-1}p' : \wedge^{i+r-1}(L) \rightarrow \wedge^{i+r-1}(L')$ and $\wedge^{j+r-1}p'' : \wedge^{j+r-1}(L) \rightarrow \wedge^{j+r-1}(L'')$ be the maps induced by the projections p' and p'' , respectively. Define FG -equivariant homomorphisms

$$D'^r : \text{Hom}_F(\wedge^i(L), \wedge^{i+r-1}(L)) \rightarrow \text{Hom}_F(\wedge^i(L'), \wedge^{i+r-1}(L'))$$

by

$$D'^r(f)(x) = \wedge^{i+r-1}p'(f(x \otimes 1)) \text{ for all } x \in \wedge^i(L),$$

and

$$D''^r : \text{Hom}_F(\wedge^j(L), \wedge^{j+r-1}(L)) \rightarrow \text{Hom}_F(\wedge^j(L''), \wedge^{j+r-1}(L''))$$

by

$$D''^r(f)(y) = \wedge^{j+r-1} p''(f(1 \otimes y)) \text{ for all } y \in \wedge^j(L'').$$

Lemma 3.4. The map D'^r is given by the composition $p'_* \circ \iota'^*$, where

$$\text{Hom}(\wedge^i(L), \wedge^{i+r-1}(L)) \xrightarrow{\iota'^*} \text{Hom}(\wedge^i(L'), \wedge^{i+r-1}(L')) \xrightarrow{p'_*} \text{Hom}(\wedge^i(L'), \wedge^{i+r-1}(L')),$$

and D''^r is the composition $p''_* \circ \iota''^*$, where

$$\text{Hom}(\wedge^j(L), \wedge^{j+r-1}(L)) \xrightarrow{\iota''^*} \text{Hom}(\wedge^j(L''), \wedge^{j+r-1}(L'')) \xrightarrow{p''_*} \text{Hom}(\wedge^j(L''), \wedge^{j+r-1}(L'')).$$

Proof. This is an immediate consequence of the definitions of D'^r and D''^r . \square

Proposition 3.5. Let $i \geq 0$ and $r \geq 2$. Suppose $E_r'^{s,t}(M) = E_2'^{s,t}(M)$ and $E_r^{s,t}(H_i(L, F)) = E_2^{s,t}(H_i(L, F))$ when $(s, t) = (0, i)$ and when $(s, t) = (r, i - r + 1)$, for $M = H_i(L', F)$ and for $M = H_i(L, F)$. Then $d_r^{0,i} \circ (C_0'^1)_* = (C_0'^r)_* \circ d_r'^{0,i}$ and $d_r'^{0,i} \circ D_*'^1 = D_*'^r \circ d_r^{0,i}$.

Proof. The first claim asserts that the following diagram commutes.

$$\begin{array}{ccc} H^0(G, H^i(L', H_i(L', F))) & \xrightarrow{d_r'^{0,i}} & H^r(G, H^{i-r+1}(L', H_i(L', F))) \\ (C_0'^1)_* \downarrow & & (C_0'^r)_* \downarrow \\ H^0(G, H^i(L, H_i(L, F))) & \xrightarrow{d_r^{0,i}} & H^r(G, H^{i-r+1}(L, H_i(L, F))) \end{array}$$

By Lemma 3.2, we know that the map $(C_0'^*)_*$ is induced by the inclusion $\iota' : L' \rightarrow L$ and the projection $p' : L \rightarrow L'$. Hence, this diagram is the outer square of the commutative diagram below.

$$\begin{array}{ccc}
H^0(G, H^i(L', H_i(L', F))) & \xrightarrow{d_r'^{0,i}} & H^r(G, H^{i-r+1}(L', H_i(L', F))) \\
\downarrow \iota'_* & & \downarrow \iota'_* \\
H^0(G, H^i(L', H_i(L, F))) & \xrightarrow{d_r'^{0,i}} & H^r(G, H^{i-r+1}(L', H_i(L, F))) \\
\downarrow p'^* & & \downarrow p'^* \\
H^0(G, H^i(L, H_i(L, F))) & \xrightarrow{d_r^{0,i}} & H^r(G, H^{i-r+1}(L, H_i(L, F)))
\end{array}$$

The second claim follows by a similar argument using Lemma 3.4. \square

As previously noted, the category $\mathcal{M}_F(L, G)$ can be identified with the category of all finite rank FG -lattices. In view of this, we will not distinguish between $\mathcal{M}_F(L', G)$, $\mathcal{M}_F(L'', G)$, and $\mathcal{M}_F(L, G)$.

Lemma 3.6. There are natural morphisms of spectral sequences $\pi'^* : E'_r \rightarrow E_r$, $\phi'^* : E_r \rightarrow E'_r$ and $\pi''^* : E''_r \rightarrow E_r$, $\phi''^* : E_r \rightarrow E''_r$, such that for all $r \geq 2$, all $s \geq 0$, and all coefficient modules in $\mathcal{M}_F(L, G)$,

$$d_r'^{s,t} = \phi'^* \circ d_r^{s,t} \circ \pi'^* \quad \text{and} \quad d_r''^{s,t} = \phi''^* \circ d_r^{s,t} \circ \pi''^*.$$

Proof. We observe that the map ι' induces a natural inclusion $\phi' : \Gamma' \rightarrow \Gamma$ and the map p' induces a natural projection $\pi' : \Gamma \rightarrow \Gamma'$ such that the composition $\pi' \circ \phi'$ is the identity map on Γ' . Let $\varphi : L \rightarrow \Gamma$ and $\varphi' : L' \rightarrow \Gamma'$ be the canonical inclusions. It follows that $\phi' \circ \varphi' = \varphi \circ \iota'$ and $\pi' \circ \varphi = \varphi' \circ p'$. Hence, the homomorphisms ϕ' and π' give rise to spectral sequence morphisms $\phi'^* : E'_* \rightarrow E_*$ and $\pi'^* : E_* \rightarrow E'_*$, such that $\phi'^* \circ \pi'^*$ is the identity morphism on $\{E'_*, d'_*\}$. Then, $d_r'^{s,t} = \phi'^* \circ \pi'^* \circ d_r^{s,t} = \phi'^* \circ d_r^{s,t} \circ \pi'^*$. Analogously, ι'' and p'' induce an inclusion $\phi'' : \Gamma'' \rightarrow \Gamma$ and a projection $\pi'' : \Gamma \rightarrow \Gamma''$, respectively, such that the composition $\pi'' \circ \phi''$ is the identity map on Γ'' and $d_r''^{s,t} = \phi''^* \circ d_r^{s,t} \circ \pi''^*$. \square

We are now ready to prove our main result.

Theorem 3.7. Recall that G is an arbitrary group, and that L is a $\mathbb{Z}G$ -lattice of finite rank that decomposes into a direct sum of the $\mathbb{Z}G$ -lattices L' and L'' . Let $r \geq 2$ and $t \geq 0$.

- (a) Suppose $v_k^i(L') = v_k^j(L'') = 0$ for all $i, j \leq t$ and for all $k < r$. Then $d_2^{s,m} = \dots = d_{r-1}^{s,m} = 0$ for all $s \geq 0$, all $m \leq t$, and all coefficient modules in $\mathcal{M}_F(L, G)$. Additionally,

$$v_r^t(L) = \sum_{i+j=t} \left((C_j'^r)_*(v_r^i(L')) + (-1)^i (C_i''^r)_*(v_r^j(L'')) \right).$$

- (b) Suppose $v_k^t(L) = 0$ for all $k < r$. Then $d_k^{s,t} = d_k''^{s,t} = 0$ for all $s \geq 0$, all $k < r$, and all coefficient modules in $\mathcal{M}_F(L, G)$. Additionally,

$$v_r^t(L') = D_*'^r(v_r^t(L)) \quad \text{and} \quad v_r^t(L'') = D_*''^r(v_r^t(L)).$$

Proof. To show part (a), we will use induction on $k \geq 2$ to prove that $v_k^m(L)$ satisfies the sum formula for all $m \leq t$ and all $k \leq r$. For each $k < r$, since $v_k^i(L') = v_k^j(L'') = 0$ for all $i, j \leq t$, this will imply $v_k^m(L) = 0$ and hence, by Theorem 2.2(b), $d_k^{s,m} = 0$ for all $s \geq 0$, all $m \leq t$, and all $M \in \mathcal{M}_F(L, G)$.

Suppose $d_2^{s,m} = \dots = d_{k-1}^{s,m} = 0$ for all $s \geq 0$, all $m \leq t$, and all $M \in \mathcal{M}_F(L, G)$. Recall that $v_k^m(L) = d_k^{0,m}([\text{id}^m])$, where $\text{id}^m \in \text{Hom}_{FG}(\wedge^m(L), \wedge^m(L))$ is the identity map. Similarly, $v_k^i(L') = d_k^{i,0}([\text{id}^i])$ and $v_k^j(L'') = d_k^{0,j}([\text{id}^j])$. Consider the decomposition $\text{id}^m = \sum_{i+j=m} \text{id}_{ij}$, where $\text{id}_{ij} : \wedge^{i+j}(L) \rightarrow \wedge^{i+j}(L)$ is the F -linear map given by

$$\text{id}_{ij}(x \otimes y) = \begin{cases} x \otimes y & \text{if } x \in \wedge^i(L') \text{ and } y \in \wedge^j(L'') \\ 0 & \text{otherwise.} \end{cases}$$

Given $i, j \geq 0$, let $x \in \wedge^i(L')$ and $y \in \wedge^j(L'')$. Then, $\text{id}_{ij}(x \otimes y) = x \otimes y = (x \otimes 1) \wedge (1 \otimes y) = C_0'^1(\text{id}^i)(x \otimes 1) \wedge C_0''^1(\text{id}^j)(1 \otimes y)$. This implies $[\text{id}_{ij}] = (C_0'^1)_*([\text{id}^i]) \cdot (C_0''^1)_*([\text{id}^j]) \in H^0(G, H^{i+j}(L, H_{i+j}(L, F)))$, and thus

$$[\text{id}^m] = \sum_{i+j=m} (C_0'^1)_*([\text{id}^i]) \cdot (C_0''^1)_*([\text{id}^j]).$$

By applying the product formula for the differentials, we compute

$$\begin{aligned}
v_k^m(L) &= d_k^{0,m} \left(\sum_{i+j=m} (C_0'^1)_*([\text{id}^i]) \cdot (C_0''^1)_*([\text{id}''^j]) \right) \\
&= \sum_{i+j=m} d_k^{0,m} \left((C_0'^1)_*([\text{id}^i]) \cdot (C_0''^1)_*([\text{id}''^j]) \right) \\
&= \sum_{i+j=m} \left(d_k^{0,i}((C_0'^1)_*([\text{id}^i])) \cdot (C_0''^1)_*([\text{id}''^j]) + (-1)^i (C_0'^1)_*([\text{id}^i]) \cdot d_k^{0,j}((C_0''^1)_*([\text{id}''^j])) \right) \\
&\stackrel{3.5}{=} \sum_{i+j=m} \left((C_0'^k)_*(d_k'^{0,i}([\text{id}^i])) \cdot (C_0''^1)_*([\text{id}''^j]) + (-1)^i (C_0'^1)_*([\text{id}^i]) \cdot (C_0''^k)_*(d_k''^{0,j}([\text{id}''^j])) \right) \\
&= \sum_{i+j=m} \left((C_0'^k)_*(v_k^i(L')) \cdot (C_0''^1)_*([\text{id}''^j]) + (-1)^i (C_0'^1)_*([\text{id}^i]) \cdot (C_0''^k)_*(v_k^j(L'')) \right) \\
&= \sum_{i+j=m} \left((C_0'^k)_*(v_k^i(L')) + (-1)^i (C_0''^k)_*(v_k^j(L'')) \right).
\end{aligned}$$

The last equality follows from the definitions of $C_*'^*$ and $C_*''^*$ and the fact that id^i and id''^j are identity maps.

To prove (b), we will use induction on $k \geq 2$ to show that $v_k^t(L') = D_*'^r(v_k^t(L))$ and $v_k^t(L'') = D_*''^r(v_k^t(L))$ for all $k \leq r$. By Theorem 2.2, for each $k < r$ and for all $s \geq 0$ this will imply that $d_k'^{s,t} = d_k''^{s,t} = 0$ for all coefficient modules in $\mathcal{M}_F(L, G)$. Note that this assertion also follows from Lemma 3.6.

For the identity map $\text{id}^t : \wedge^t(L') \rightarrow \wedge^t(L')$, we have $\text{id}^t(x) = x = \wedge^t p'(x \otimes 1) = \wedge^t p'(\text{id}^t(x \otimes 1)) = D_*'^1(\text{id}^t)(x)$ for all $x \in \wedge^t(L')$. This implies $[\text{id}^t] = D_*'^1([\text{id}^t]) \in E_k'^{0,t}(H_t(L', F))$. Thus, it follows

$$v_k^t(L') = d_k'^{0,t}([\text{id}^t]) = d_k'^{0,t} \circ D_*'^1([\text{id}^t]) \stackrel{3.5}{=} D_*'^k \circ d_k'^{0,t}([\text{id}^t]) = D_*'^k(v_k^t(L)).$$

By an analogous argument, $v_k^t(L'') = D_*''^k(v_k^t(L))$. □

Remark 3.8. Using the same assumptions as in Theorem 3.7(a), the sum formula can be simplified. Since $E_r'^{r,i-r+1} = E_r''^{r,j-r+1} = 0$, $v_r^i(L') = v_r^j(L'') = 0$ holds a priori for all $i, j < r - 1$. It is an easy exercise to check that $d_r'^{0,r-1} = 0$ and $d_r''^{0,r-1} = 0$ for all $M \in \mathcal{M}_F(L, G)$, since they are differentials with target in the 0th row (see Prop.1, [3]).

Therefore, $v_r^{r-1}(L') = v_r^{r-1}(L'') = 0$ and we have

$$\begin{aligned} v_r^t(L) &= \sum_{i+j=t} \left((C_j'^r)_*(v_r^i(L')) + (-1)^i (C_i''^r)_*(v_r^j(L'')) \right) \\ &= \sum_{i+j=t-r} (C_j'^r)_*(v_r^{i+r}(L')) + \sum_{i+j=t-r} (-1)^i (C_i''^r)_*(v_r^{j+r}(L'')) \\ &= \sum_{i+j=t-r} \left((C_j'^r)_*(v_r^{i+r}(L')) + (-1)^i (C_i''^r)_*(v_r^{j+r}(L'')) \right). \end{aligned}$$

4. SOME COROLLARIES

Theorem 3.7 has particularly interesting applications if characteristic classes of the $\mathbb{Z}G$ -lattices L' , L'' , and L are viewed as obstructions to the vanishing of the differentials in the associated Lyndon-Hochschild-Serre spectral sequences. We use the same notation as before.

Corollary 4.1. Let $r \geq 2$. If $v_k^i(L') = v_k^j(L'') = 0$ for all $i, j \in [k, r]$ and for all $k < r$, then $d_2^{s,m} = \dots = d_{r-1}^{s,m} = 0$ for all $s \geq 0$, all $m \leq r$, and all $M \in \mathcal{M}_F(L, G)$. Moreover,

$$v_r^r(L) = (C_0'^r)_*(v_r^r(L')) + (C_0''^r)_*(v_r^r(L'')).$$

Proof. This is a direct consequence of Theorem 3.7 and the preceding remark. \square

A consequence of Lemma 3.6 is the fact that when the differentials in the Lyndon-Hochschild-Serre spectral sequence $\{E_*, d_*\}$ are all zero, the same is true for the spectral sequences corresponding to the $\mathbb{Z}G$ -sublattices L' and L'' . The next corollary gives us a converse.

Corollary 4.2. Suppose $v_p^t(L') = v_q^t(L'') = 0$ for all $p \leq \dim(L')$, all $q \leq \dim(L'')$, and all $t \geq 0$. Then $d_k^{s,t} = 0$ for all $M \in \mathcal{M}_F(L, G)$, all $s, t \geq 0$, and all $k \geq 2$. Moreover, if in addition $\{E_*, d_*\}$ has no extension problems, then for every $n \geq 0$ and for all $M \in \mathcal{M}_F(L, G)$ we have

$$H^n(\Gamma, M) = \bigoplus_{i+j=n} H^i(G, H^j(L, M)).$$

Proof. Since $v_p^t(L') = d_p^{0,t}([\text{id}^t]) \in E_p^{p,t-p+1}(H_t(L', F))$, this class lies in the image of the map

$$d_p^{0,t} : H^0(G, H^t(L', H_t(L', F))) \rightarrow H^p(G, H^{t-p+1}(L', H_t(L', F))).$$

Note that $d_p^{0,t} = 0$ when $t > \dim(L')$ or $p > t + 1$. If $p = t + 1$, then $d_{t+1}^{0,t} = 0$, since it is a differential with a target in the 0th row (see Prop.1, [3]). Therefore, if $v_p^*(L') = 0$ for all $p \leq \dim(L')$, then all characteristic classes of the spectral sequence $\{E'_*, d'_*\}$ are zero. A similar argument shows that all characteristic classes of $\{E''_*, d''_*\}$ are zero when $v_q^*(L'') = 0$ for all $q \leq \dim(L'')$. \square

Corollary 4.3. Let $t \geq 0$. Set $\Gamma' = L' \rtimes G$ and $\Gamma'' = L'' \rtimes G$. Let $\varphi' : L' \rightarrow \Gamma'$, $\varphi'' : L'' \rightarrow \Gamma''$, and $\varphi : L \rightarrow \Gamma$ be the natural inclusions.

- (a) If $\varphi'^* : H^m(\Gamma', H_m(L', F)) \rightarrow H^m(L', H_m(L', F))^G$ and $\varphi''^* : H^m(\Gamma'', H_m(L'', F)) \rightarrow H^m(L'', H_m(L'', F))^G$ are surjective for all $m \leq t$, then $\varphi^* : H^m(\Gamma, M) \rightarrow H^m(L, M)^G$ is surjective for all $m \leq t$ and for all $M \in \mathcal{M}_F(L, G)$.
- (b) If $\varphi^* : H^t(\Gamma, M) \rightarrow H^t(L, M)^G$ is surjective, then $\varphi'^* : H^t(\Gamma', M) \rightarrow H^t(L', M)^G$ and $\varphi''^* : H^t(\Gamma'', M) \rightarrow H^t(L'', M)^G$ are surjective for all $M \in \mathcal{M}_F(L, G)$.

Proof. To prove (a), we observe that Corollary 2.4 implies $v_r^i(L') = v_r^j(L'') = 0$ for all $r \geq 0$ and for all $i, j \leq t$. Then, by Theorem 3.7, $d_r^{s,m} = 0$ for all $r, s \geq 0$, all $m \leq t$, and all coefficient modules in $\mathcal{M}_F(L, G)$. Applying again Corollary 2.4 finishes the proof.

For part (b), let $\iota'^* : H^t(L, M)^G \rightarrow H^t(L', M)^G$, $\phi'^* : H^t(\Gamma, M) \rightarrow H^t(\Gamma', M)$, and $p'^* : H^t(L', M)^G \rightarrow H^t(L, M)^G$ be the induced maps of the inclusions $\iota' : L' \rightarrow L$, $\phi' : \Gamma' \rightarrow \Gamma$, and the projection $p' : L \rightarrow L'$, respectively. Then, we have $\varphi'^* \circ \phi'^* = \iota'^* \circ \varphi^*$. Since $\iota'^* \circ p'^*$ is the identity map on $H^t(L', M)^G$, ι'^* is surjective and hence $\iota'^* \circ \varphi^*$ is surjective. The previous equality shows that φ'^* is also surjective. Similarly, it follows that φ''^* is surjective. \square

Given an arbitrary finite rank integral $\mathbb{Z}G$ -lattice L , Lieberman's result (Thm.4, [3]) states that in the associated Lyndon-Hochschild-Serre spectral sequence $\{E_*, d_*\}$, for any $s, t \geq 0$ and $r \geq 2$, the image of the differential $d_r^{s,t}$ is a torsion group annihilated by the integers $m^{t-r+1}(m^{r-1} - 1)$ for all $m \in \mathbb{Z}$. Using this fact, it was proved in [3] that if F is a field of nonzero characteristic p , then $d_r^{s,t} = 0$ for all $s, t \geq 0$ and all $r < p$. The next corollary follows from combining this result with the sum formula of Remark 3.8.

Corollary 4.4. Let F be a field of nonzero characteristic p . Assume L' and L'' are $\mathbb{Z}G$ -lattices of finite rank and $L = L' \oplus L''$.

- (a) $d_r^{s,t} = 0$ for all $s, t \geq 0$, all $r < p$, and all $M \in \mathcal{M}_F(L, G)$.

$$(b) \quad d_p^{s,t}(x) = (-1)^s y \cdot \sum_{i+j=t-p} \left((C_j'^p)_*(v_p^{i+p}(L')) + (-1)^i (C_i''^p)_*(v_p^{j+p}(L'')) \right)$$

for all $M \in \mathcal{M}_F(L, G)$, for all $x \in E_p^{s,t}(M)$, and for all $y \in E_p^{s,0}(H^t(L, M))$ such that $\theta(y) = x$.

□

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